3

LINEAR PROGRAMMING: A GEOMETRIC APPROACH

3.4 Sensitivity Analysis

Sensitivity Analysis

In this section, we investigate how changes in the parameters of a linear programming problem affect its optimal solution. This type of analysis is called sensitivity analysis.

Consider the following objective function and constraints.

Maximize $P = x + 1.2y$ — Objective function

subject to $2x + y \leq 180$ — Constraint 1

$x + 3y \leq 300$ — Constraint 2

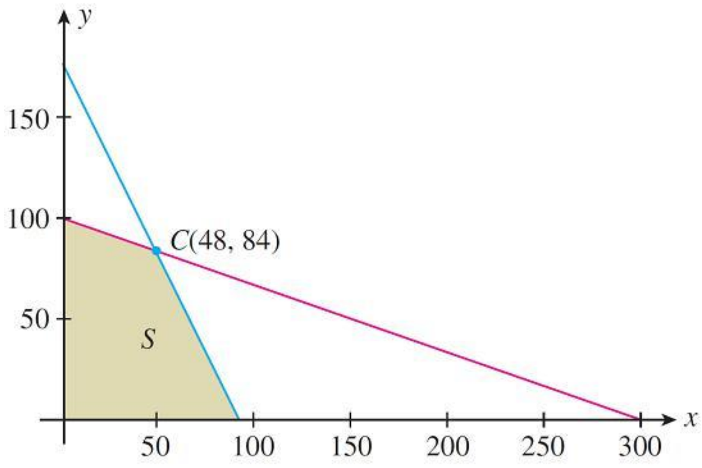
$x \geq 0, y \geq 0$

where x denotes the number of Type *A* souvenirs and y denotes the number of Type *B* souvenirs to be made.

Sensitivity Analysis

The optimal solution of this problem is $x = 48$, $y = 84$ (corresponding to the point C).

The optimal value of P is 148.8 (Figure 17).



The optimal solution occurs at the point $C(48, 84)$.

Figure 17

Sensitivity Analysis

The following questions arise in connection with this production problem.

1. How do changes made to the coefficients of the objective function affect the optimal solution?

2. How do changes made to the constants on the right-hand side of the constraints affect the optimal solution?

Changes in the Coefficients of the Objective Function

Changes in the Coefficients of the Objective Function

In the production problem under consideration, the objective function is $P = x + 1.2y$.

The coefficient of x, which is 1, tells us that the contribution to the profit for each Type *A* souvenir is \$1.00. The coefficient of y, 1.2, tells us that the contribution to the profit for each Type *B* souvenir is \$1.20.

Now suppose the contribution to the profit for each Type *B* souvenir remains fixed at \$1.20 per souvenir. By how much can the contribution to the profit for each Type *A* souvenir vary without affecting the current optimal solution?

Changes in the Coefficients of the Objective Function

To answer this question, suppose the contribution to the profit of each Type *A* souvenir is \$*c* so that

$$P = cx + 1.2y \tag{8}$$

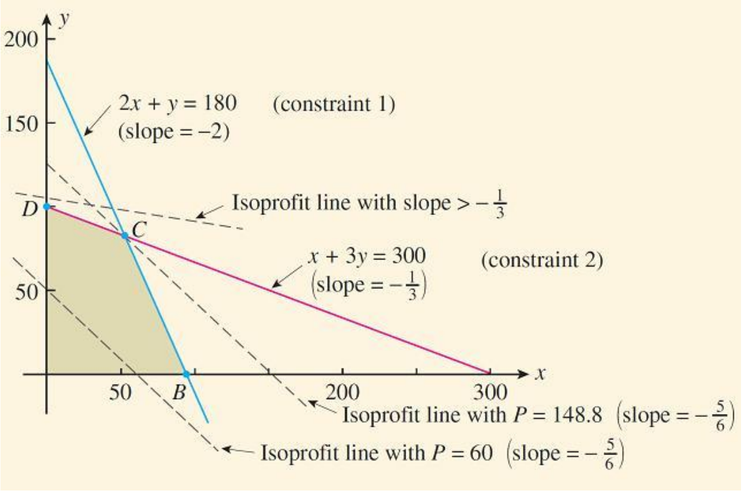
We need to determine the range of values of *c* such that the solution remains optimal.

We begin by rewriting Equation (8) for the isoprofit line in the slope-intercept form. Thus,

$$y = -\frac{c}{1.2}x + \frac{P}{1.2} \tag{9}$$

Changes in the Coefficients of the Objective Function

The slope of the isoprofit line is $-c/1.2$. If the slope of the isoprofit line exceeds that of the line associated with constraint 2, then the optimal solution shifts from point *C* to point *D* (Figure 18).



Increasing the slope of the isoprofit line $P = cx + 1.2y$ beyond $-\frac{1}{3}$ shifts the optimal solution from point *C* to point *D*.

Figure 18

Changes in the Coefficients of the Objective Function

On the other hand, if the slope of the isoprofit line is *less than or equal to* the slope of the line associated with constraint 2, then the optimal solution remains unaffected. (You may verify $-\frac{1}{3}$ that is the slope of the line associated with constraint 2 by writing the equation $x + 3y = 300$ in the slope-intercept form.) In other words, we must have

$$-\frac{c}{1.2} \leq -\frac{1}{3}$$

$$\frac{c}{1.2} \geq \frac{1}{3}$$

Multiplying each side by –1 reverses the inequality sign.

$$c \geq \frac{1.2}{3} = 0.4$$

Changes in the Coefficients of the Objective Function

A similar analysis shows that if the slope of the isoprofit line is less than that of the line associated with constraint 1, then the optimal solution shifts from point *C* to point *B*.

Since the slope of the line associated with constraint 1 is –2, we see that point *C* will remain optimal provided that the slope of the isoprofit line is *greater than or equal to* –2, that is, if

$$-\frac{c}{1.2} \geq -2$$

$$\frac{c}{1.2} \leq 2$$

$$c \leq 2.4$$

Changes in the Coefficients of the Objective Function

Thus, we have shown that if $0.4 \leq c \leq 2.4$, then the optimal solution that we obtained previously remains unaffected.

This result tells us that if the contribution to the profit of each Type *A* souvenir lies between $0.40 and $2.40, then Ace Novelty should still make 48 Type *A* souvenirs and 84 Type *B* souvenirs.

Of course, the company's profit will change with a change in the value of *c*—it's the product mix that stays the same.

Changes in the Coefficients of the Objective Function

For example, if the contribution to the profit of a Type *A* souvenir is $1.50, then the company's profit will be $172.80.

Incidentally, our analysis shows that the parameter *c* is not a sensitive parameter.

Applied Example 1 – *Profit Function Analysis*

Kane Manufacturing has a division that produces two models of grates, model A and model B.

To produce each model A grate requires 3 pounds of cast iron and 6 minutes of labor.

To produce each model B grate requires 4 pounds of cast iron and 3 minutes of labor.

The profit for each model A grate is $2.00, and the profit for each model B grate is $1.50. Available for grate production each day are 1000 pounds of cast iron and 20 labor-hours.

Applied Example 1 – *Profit Function Analysis*

Because of an excess inventory of model A grates, management has decided to limit the production of model A grates to no more than 180 grates per day.

a. Use the method of corners to determine the number of grates of each model Kane should produce in order to maximize its profit.

Applied Example 1 – *Profit Function Analysis*

b. Find the range of values that the contribution to the profit of a model A grate can assume without changing the optimal solution.

c. Find the range of values that the contribution to the profit of a model B grate can assume without changing the optimal solution.

Applied Example 1(a) – *Solution*

Let x denote the number of model A grates produced, and let y denote the number of model B grates produced. Then verify that we are led to the following linear programming problem:

$$\text{Maximize } P = 2x + 1.5y$$

$$\text{subject to } 3x + 4y \le 1000 \quad \text{Constraint 1}$$

$$6x + 3y \le 1200 \quad \text{Constraint 2}$$

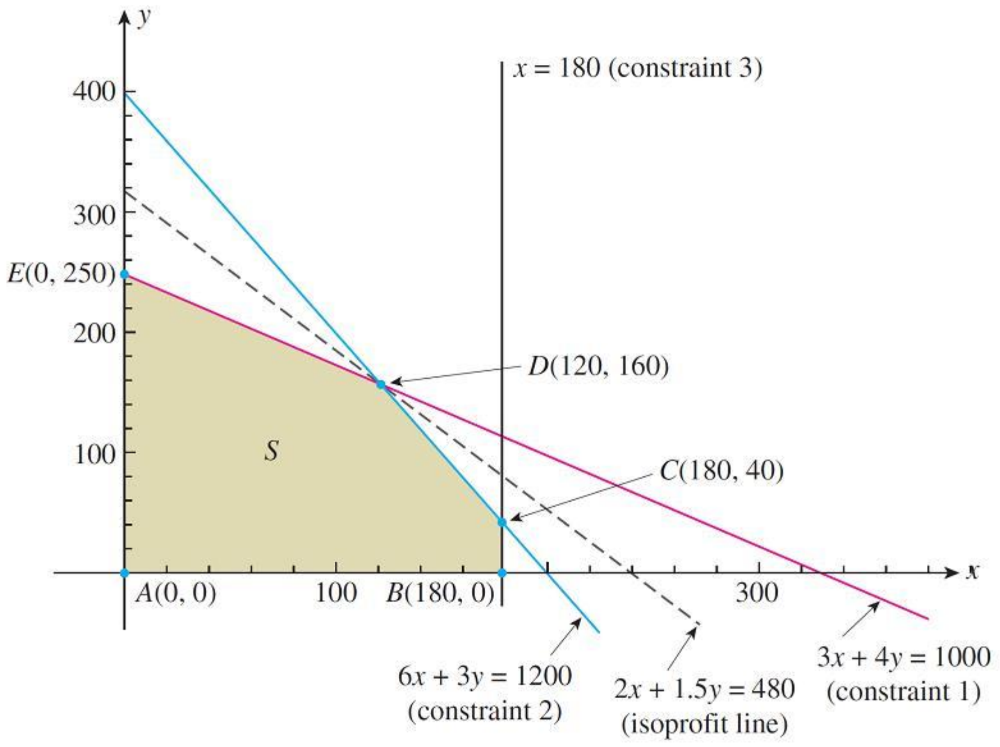
$$x \le 180 \quad \text{Constraint 3}$$

$$x \ge 0,\ y \ge 0$$

Applied Example 1(a) – *Solution*

cont'd

The graph of the feasible set S is shown in Figure 19.



The shaded region is the feasible set S. Also shown are the lines of the equations associated with the constraints.

Figure 19

Applied Example 1(a) – *Solution* cont’d

From the following table of values,

Vertex	$P = 2x + 1.5y$
$A(0, 0)$	0
$B(180, 0)$	360
$C(180, 40)$	420
$D(120, 160)$	480
$E(0, 250)$	375

we see that the maximum of $P = 2x + 1.5y$ occurs at the vertex $D(120, 160)$ with a value of 480. Thus, Kane realizes a maximum profit of $480 per day by producing 120 model A grates and 160 model B grates each day.

Applied Example 1(b) – *Solution* cont'd

Let *c* (in dollars) denote the contribution to the profit of a model A grate. Then *P* = *cx* + 1.5*y* or, upon solving for *y*,

$$y = -\frac{c}{1.5}x + \frac{P}{1.5}$$

$$= \left(-\frac{2}{3}c\right)x + \frac{2}{3}P$$

Referring to Figure 19, you can see that if the slope of the isoprofit line is greater than the slope of the line associated with constraint 1, then the optimal solution will shift from point *D* to point *E*.

Applied Example 1(b) – *Solution* cont'd

Thus, for the optimal solution to remain unaffected, the slope of the isoprofit line must be less than or equal to the slope of the line associated with constraint 1.

But the slope of the line associated with constraint 1 is $-\frac{3}{4}$ which you can see by rewriting the equation $3x + 4y = 1000$ in the slope-intercept form $y = -\frac{3}{4}x + 250$.

Applied Example 1(b) – *Solution*

cont'd

Since the slope of the isoprofit line is –2*c*/3, we must have

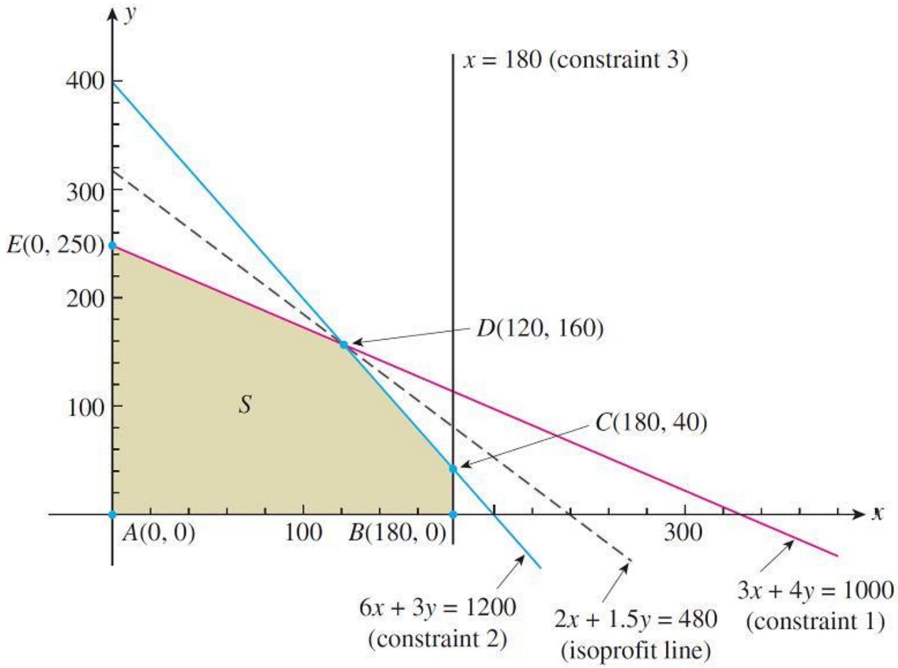
$$-\frac{2c}{3} \leq -\frac{3}{4}$$

$$\frac{2c}{3} \geq \frac{3}{4}$$

$$c \geq \left(\frac{3}{4}\right)\left(\frac{3}{2}\right) = \frac{9}{8} = 1.125$$

Applied Example 1(b) – *Solution* cont'd

Again referring to Figure 19, you can see that if the slope of the isoprofit line is less than that of the line associated with constraint 2, then the optimal solution shifts from point *D* to point *C*.



The shaded region is the feasible set *S*. Also shown are the lines of the equations associated with the constraints.

Figure 19

Applied Example 1(b) – *Solution* cont'd

Since the slope of the line associated with constraint 2 is –2 (rewrite the equation $6x + 3y = 1200$ in the slope-intercept form

$$y = -2x + 400),$$

we see that the optimal solution remains at point *D* provided that the slope of the isoprofit line is greater than or equal to –2; that is,

$$-\frac{2c}{3} \geq -2$$

$$\frac{2c}{3} \leq 2$$

$$c \leq (2)\left(\frac{3}{2}\right) = 3$$

Applied Example 1(b) – *Solution* cont'd

We conclude that the contribution to the profit of a model A grate can assume values between $1.125 and $3.00 without changing the optimal solution.

Applied Example 1(c) – *Solution* cont'd

Let c (in dollars) denote the contribution to the profit of a model B grate. Then

$$P = 2x + cy$$

or, upon solving for y,

$$y = -\frac{2}{c}x + \frac{P}{c}$$

Applied Example 1(c) – *Solution*

cont'd

An analysis similar to that performed in part (b) with respect to constraint 1 shows that the optimal solution will remain in effect provided that

$$-\frac{2}{c} \leq -\frac{3}{4}$$

$$\frac{2}{c} \geq \frac{3}{4}$$

$$c \leq 2\left(\frac{4}{3}\right) = \frac{8}{3} = 2\frac{2}{3}$$

Applied Example 1(c) – *Solution* cont'd

Performing an analysis with respect to constraint 2 shows that the optimal solution will remain in effect provided that

$$-\frac{2}{c} \geq -2$$

$$\frac{2}{c} \leq 2$$

$$c \geq 1$$

Thus, the contribution to the profit of a model B grate can assume values between $1.00 and $2.67 without changing the optimal solution.

Changes to the Constants on the Right-Hand Side of the Constraint Inequalities

Let's consider the production problem:

$$\text{Maximize } P = x + 1.2y$$

$$\text{subject to } 2x + y \leq 180 \quad \text{Constraint 1}$$

$$x + 3y \leq 300 \quad \text{Constraint 2}$$

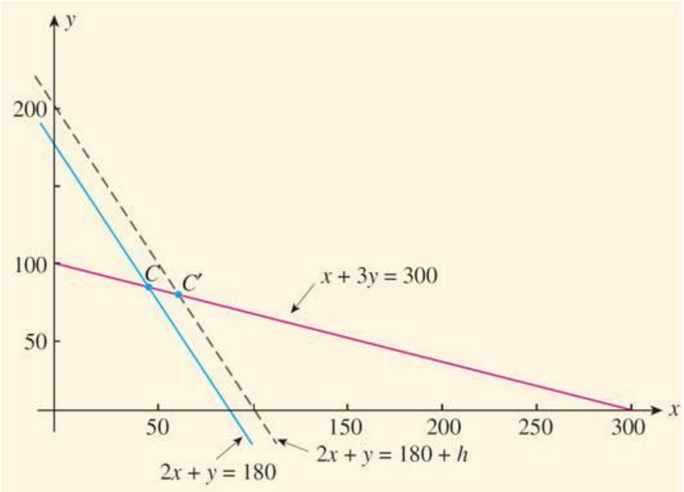
$$x \geq 0,\ y \geq 0$$

Now suppose that the time available on Machine I is changed from 180 minutes to (180 + h) minutes, where h is a real number. Then the constraint on Machine I is changed to

$$2x + y \leq 180 + h$$

Observe that the line with equation $2x + y = 180 + h$ is parallel to the line $2x + y = 180$ associated with the original constraint 1.

As you can see from Figure 20, the result of adding the constant *h* to the right-hand side of constraint 1 is to shift the current optimal solution from the point *C* to the new optimal solution occurring at the point *C*′.



The lines with equations $2x + y = 180$ and $2x + y = 180 + h$ are parallel to each other.

Figure 20

To find the coordinates of C', we observe that C' is the point of intersection of the lines with equations

$$2x + y = 180 + h \text{ and } x + 3y = 300$$

Thus, the coordinates of the point are found by solving the system of linear equations

$$2x + y = 180 + h$$

$$x + 3y = 300$$

The solutions are

$$x = \frac{3}{5}(80 + h) \quad \text{and} \quad y = \frac{1}{5}(420 - h) \qquad \textbf{(10)}$$

The nonnegativity of *x* implies that

$$\frac{3}{5}(80 + h) \geq 0$$

$$80 + h \geq 0$$

$$h \geq -80$$

Next, the nonnegativity of *y* implies that

$$\frac{1}{5}(420 - h) \geq 0$$

$$420 - h \geq 0$$

$$h \leq 420$$

Thus, *h* must satisfy the inequalities $-80 \leq h \leq 420$. Our computations reveal that a meaningful solution will require that the time available for Machine I must range between (180 – 80) and (180 + 420) minutes—that is, between 100 and 600 minutes.

Changes to the Constants on the Right-Hand Side of the Constraint Inequalities

Under these conditions, Ace Novelty should produce $\frac{3}{5}(80 + h)$ Type *A* souvenirs and $\frac{1}{5}(420 - h)$ Type *B* souvenirs.

For example, if Ace Novelty can manage to increase the time available on Machine I by 10 minutes, then it should produce $\frac{3}{5}(80 + 10)$, or 54, Type *A* souvenirs and $\frac{1}{5}(420 - 10)$, or 82, Type *B* souvenirs; the resulting profit is

$$P = x + 1.2y = 54 + (1.2)(82) = 152.4$$

or $152.40.

We leave it as an exercise for you to show that if the time available on Machine II is changed from 300 minutes to (300 + *k*) minutes with no change in the maximum capacity for Machine I, then *k* must satisfy the inequalities
$-210 \le k \le 240$.

Thus, for a meaningful solution to the problem, the time available on Machine II must lie between 90 and 540 min. Furthermore, in this case, Ace Novelty should produce $\frac{1}{5}(240 - k)$ Type *A* souvenirs and $\frac{1}{5}(420 + 2k)$ Type *B* souvenirs.

Shadow Prices

Shadow Prices

We have just seen that if Ace Novelty could increase the maximum available time on Machine I by 10 minutes, then the profit would increase from the original optimal value of $148.80 to $152.40. In this case, finding the extra time on Machine I proved beneficial to the company.

More generally, to study the economic benefits that can be derived from increasing its resources, a company looks at the shadow prices associated with the respective resources.

Shadow Prices

We define the shadow price for the *i*th resource (associated with the *i*th constraint of the linear programming problem) to be the amount by which the value of the objective function is improved—increased in a maximization problem and decreased in a minimization problem—if the right-hand side of the *i*th constraint is changed by 1 unit.

In the Ace Novelty example discussed earlier, we showed that if the right-hand side of constraint 1 is increased by *h* units, then the optimal solution is given by Equations (10):

$$x = \frac{3}{5}(80 + h) \quad \text{and} \quad y = \frac{1}{5}(420 - h)$$

Shadow Prices

The resulting profit is calculated as follows:

$$P = x + 1.2y$$

$$= x + \frac{6}{5}y$$

$$= \frac{3}{5}(80 + h) + \left(\frac{6}{5}\right)\left(\frac{1}{5}\right)(420 - h)$$

$$= \frac{3}{25}(1240 + 3h)$$

Shadow Prices

Upon setting h = 1, we find

$$P = \frac{3}{25}(1240 + 3)$$

$$= 149.16$$

Since the optimal profit for the original problem is $148.80, we see that the shadow price for the first resource is 149.16 – 148.80, or $0.36.

To summarize, Ace Novelty’s profit increases at the rate of $0.36 per 1-minute increase in the time available on Machine I.

Applied Example 2 – *Shadow Prices*

Consider the problem posed in Example 1:

Maximize $P = 2x + 1.5y$

subject to $3x + 4y \leq 1000$ Constraint 1

$6x + 3y \leq 1200$ Constraint 2

$x \leq 180$ Constraint 3

$x \geq 0, y \geq 0$

Applied Example 2 – *Shadow Prices* cont'd

a. Find the range of values that Resource 1 (the constant on the right-hand side of constraint 1) can assume.

b. Find the shadow price for Resource 1.

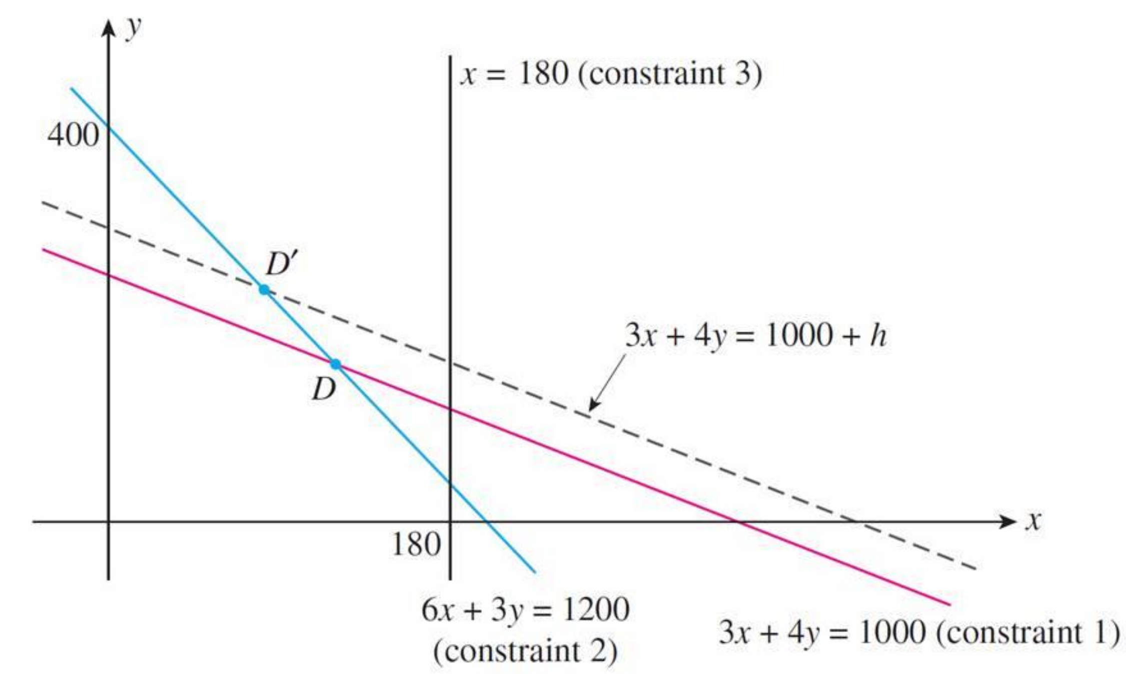
Solution:

a. Suppose the right-hand side of constraint 1 is replaced by 1000 + *h*, where *h* is a real number.

Example 2 – *Solution*

cont'd

Then the new optimal solution occurs at the point D' (Figure 21).



As the amount of Resource 1 changes, the point at which the optimal solution occurs shifts from *D* to *D′*.

Figure 21

Example 2 – *Solution*

cont'd

To find the coordinates of D', we solve the system

$$3x + 4y = 1000 + h$$

$$6x + 3y = 1200$$

Multiplying the first equation by –2 and then adding the resulting equation to the second equation, we obtain

$$-5y = -800 - 2h$$

$$y = \frac{2}{5}(400 + h)$$

$$\begin{aligned} -6x - 8y &= -2000 - 2h \\ 6x + 3y &= 1200 \\ \hline -5y &= -800 - 2h \end{aligned}$$

Example 2 – *Solution*

cont'd

Substituting this value of *y* into the second equation in the system gives

$$6x + \frac{6}{5}(400 + h) = 1200$$

$$x + \frac{1}{5}(400 + h) = 200$$

$$x = \frac{1}{5}(600 - h)$$

Example 2 – *Solution*

cont'd

The nonnegativity of y implies that $h \geq -400$, and the nonnegativity of x implies that $h \leq 600$. But constraint 3 dictates that x must also satisfy

$$x = \frac{1}{5}(600 - h) \leq 180$$

$$600 - h \leq 900$$

$$-h \leq 300$$

$$h \geq -300$$

Therefore, h must satisfy $-300 \leq h \leq 600$.

Example 2 – *Solution*

cont'd

This tells us that the amount of Resource 1 must lie between 1000 – 300, or 700, and 1000 + 600, or 1600—that is, between 700 and 1600 pounds.

b. If we set *h* = 1 in part (a), we obtain

$$x = \frac{1}{5}(600 - 1) = \frac{599}{5}$$

$$y = \frac{2}{5}(400 + 1) = \frac{802}{5}$$

Example 2 – *Solution*

cont'd

Therefore, the profit realized at this level of production is

$$P = 2x + \frac{3}{2}y = 2\left(\frac{599}{5}\right) + \frac{3}{2}\left(\frac{802}{5}\right)$$

$$= \frac{2401}{5} = 480.2$$

Since the original optimal profit is $480 (see Example 1), we see that the shadow price for Resource 1 is $0.20.

Shadow Prices

Constraints 1 and 2, which *hold with equality* at the optimal solution *D*(120, 160), are said to be binding constraints.

The objective function cannot be increased without increasing these resources. They have *positive* shadow prices.

Importance of Sensitivity Analysis

Importance of Sensitivity Analysis

We conclude this section by pointing out the importance of sensitivity analysis in solving real-world problems. The values of the parameters in these problems may change.

For example, the management of Ace Novelty might wish to increase the price of a Type *A* souvenir because of increased demand for the product, or they might want to see how a change in the time available on Machine I affects the (optimal) profit of the company.

When a parameter of a linear programming problem is changed, it is true that one need only re-solve the problem to obtain a new solution to the problem.

Importance of Sensitivity Analysis

But since a real-world linear programming problem often involves thousands of parameters, the amount of work involved in finding a new solution is prohibitive.

Another disadvantage in using this approach is that it often takes many trials with different values of a parameter to see their effect on the optimal solution of the problem.

Thus, a more analytical approach such as that discussed in this section is desirable.